VAULTED ARCHITECTURE

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CONTENTS

ABSTRACT. This paper contains two separate, complementary portions, both related to the form and structure of vaulted architecture. The first part is a history of vaulted architecture in Ancient Rome, the civilization that pioneered its widespread use and monumentalization, especially through the use of a Roman innovation: concrete. The second part of this paper contains a brief, self-contained discussion of arches and domes from a physical and mathematical point of view. Four separate configurations for arched and domed architecture are considered, and differential equations set up for the curves describing their shape. Closed-form solutions are obtained for all four cases.

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Part 1

A Brief Overview of the Techniques of Vaulted Architecture in Rome

1. Introduction

Vaulted architecture—and its related forms, the arch and the dome—represent some of Ancient Rome's greatest contributions to the modern world. The impressive accomplishments of Roman engineering and architecture in antiquity are only matched by the lasting power of Roman monumental architecture; we see it still in the Pantheon of Hadrian, the massive Basilica of Maxentius, and in the Markets of Trajan. And while the concept of the vault was not in itself a Roman invention, the Romans did, through their own advances in the fields of architecture, engineering, and construction technique, perfect the vault in its various forms.

2. The Evolution of the Roman Vault

The Romans were perhaps the first builders in Europe to fully appreciate the versatility and advantages of the arch, vault and dome (Robertson 1929, p. 231). However, they were, by no means the first developers of these structures. In fact, vaulting and arches had been used to great effect in the Near East and Egypt for centuries before the Romans (Robertson 1929, p. 231; Sear 2002, p. 17). We see arches and vaulting establishing a firm footing in Greek Hellenistic and Etruscan architecture in the fourth century BCE (Sear 2002, p. 17; Adam 2005, p. 164), though the earliest examples we see in Etruria are not semicircular arches, but rather polygonal, corbelled ones.

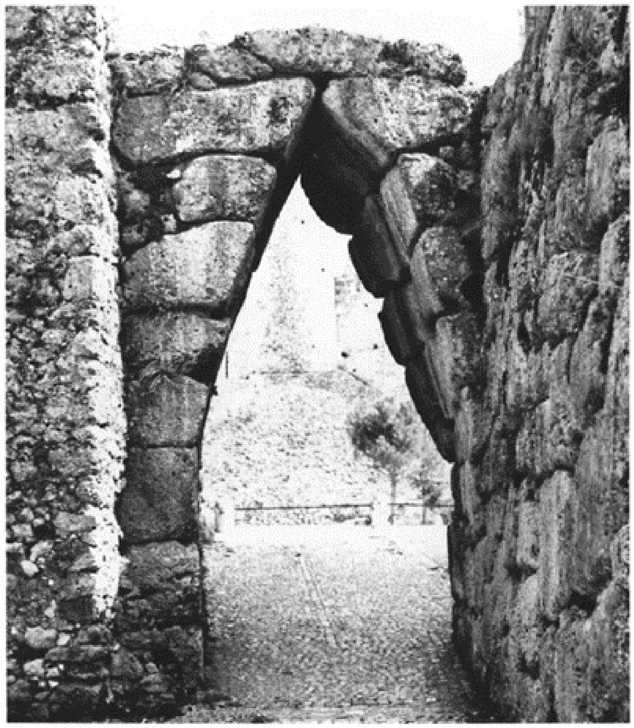

FIGURE 1. Early corbelled archway, southern Latium, fifth century BCE (Adam 2005, p. 332).

The earliest dome-like structure that we can see in the region is the famed Tholos of Atreus of Mycenae, dated to the middle of the 13th century BCE. The Greek colonies of Magna Graecia are home to the earliest archways and barrel vaults in Italy (Ward-Perkins 1977, p. 28). These were constructed out of stone, with the occasional use of lime mortar.

The general theory is that the earliest Italic archways were the work of the Etruscans, and that it was a building technique henceforth appropriated by the Latins. Indeed, the earliest archway in Rome proper was considered to be that about the outflow of the Cloaca Maxima at the Tiber. This drainage system had its genesis under the Etruscan Tarquins (and hence, Etruscan engineers) of the late sixth century BCE, and accordingly, the archway present at its outflow is attributed to this period (although this assertion has been challenged, and the archway alternatively dated to an Augustan renovation of the drain) (Adam 2005, p. 158).

FIGURE 2. Drain of the Cloaca Maxima, Forum Boarium, Rome, either ca. 509 CE or Augustan (Adam 2005, p. 321).

We otherwise see early "true" or *voussoir* archways in the Etruscan towns of Velathri, Aperusia, and Falerii Novi. It is likely that the latter, which was in fact a new Roman town built for the purpose of re-inhabiting the conquered inhabitants of the Etruscan city Falerii Veteres, was built using Etruscan engineers. The town and its archways date to approximately 241 BCE, while the structures at Velathri and Aperusia likewise date to the middle of the third century BCE (Adam 2005, p. 159). *Voussoir* arches are formed by an arc of separate stones, with a keystone at the top of the arch. The arches and surrounding walls were stabilized by utilizing rocks of varying size.

Following the Roman appropriation of the *voussoir* arch from the Etruscans, we see numerous examples of masonry arches can be found in Rome (e.g. the PONS ÆMILIUS (142 BCE), the PONS MULVIUS (109 BCE) amongst others). Once, however, traditional masonry arches were eschewed in favor of concrete ones, we are able to see an expansion of the arched form to a much larger scale.

2.1. **The Development of Concrete.** Roman concrete, or OPUS CÆMENTICUM was a mixture of mortar with stone aggregate (CÆMENTA) (Lancaster 2005, p. 3). CÆMENTA were generally about the size of a human fist, though a great deal of variation is naturally present (Sear 2002, p. 73)[1]. This formula in itself was not new and had been used to varying levels of success by the Greeks, who used a mortar comprised

[1] OPUS CÆMENTICA themselves were often classified according to the type of masonry "facing" that contained the mixture; for more information, please see Sear, pp.74-75.

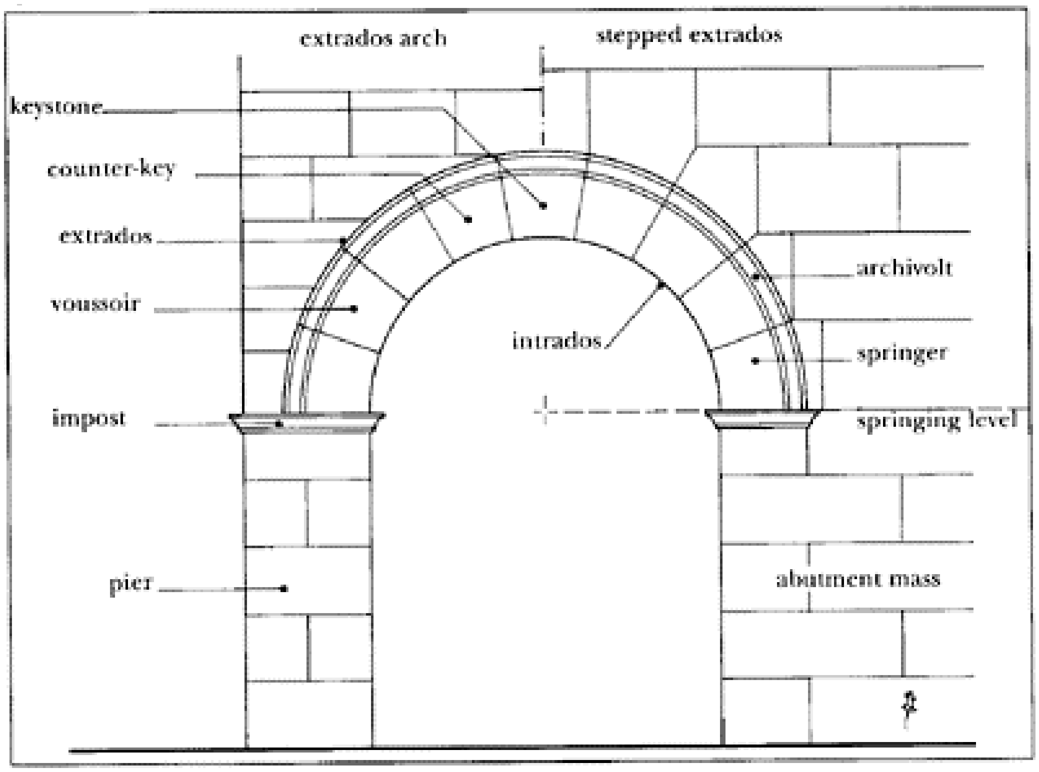

FIGURE 3. A schematic of the *voussoir*, or true, arch (Adam 2005, p. 342).

primarily of lime and water. However, the addition of local, "*pozzolana*"[2] volcanic ash to the mixture represented a breakthrough in the creation of binding mortar. *Pozzolana*, which is native to the areas surrounding Mount Vesuvius in Campania, possesses a unique chemical composition that gives the resulting mortar a substantial compressive strength (between 5 and 8 times the strength of the pure lime mortar used by the Greeks) and allows it to set even in aqueous environments (Lancaster 2005, p. 51). Moreover, the composition of the CÆMENTA was highly

[2]From Puteolana

varied by the Romans, to an effect that will be discussed in several examples later.

This pozzolana mortar was used as early as the late 3rd century BCE, though only for walls. The earliest examples of its use in vaulting occur at the Sanctuary of Fortuna at Præneste and the PORTICUS ÆMILIA, both of which are dated to the early second century BCE (Lancaster 2005, p. 11; Sear 2002, p. 19). Both the Sanctuary of Fortuna and the PORTICUS ÆMILIA are massive structures; the PORTICUS ÆMILIA, a large warehouse located on the eastern bank of the Tiber in south Rome, is a long hall measuring nearly 490 meters long and 60 meters wide, with large barrel vaults running the length of the building and smaller archways traversing its width (Sear 2002, p. 19). These early expressions of concrete vaults display a relative unfamiliarity with the material; for instance, the concrete vaults on the Terraza della Cortina at Praeneste demonstrate substantial deformation, or "creep", indicating that the builders assumed that the concrete was in fact stronger than it was (Lancaster 2005, p. 10).

Roman concrete, unlike modern concrete, was not poured, but rather the mortar and CÆMENTA laid separately by trowel (Lancaster 2005, p. 3). Nonetheless, it is with this innovation that we see the increasingly widespread employ of the masonry arch and barrel vault in Roman Italy by the middle of the second century BCE make way for the monumental concrete structures that would be characteristic of Roman architecture.

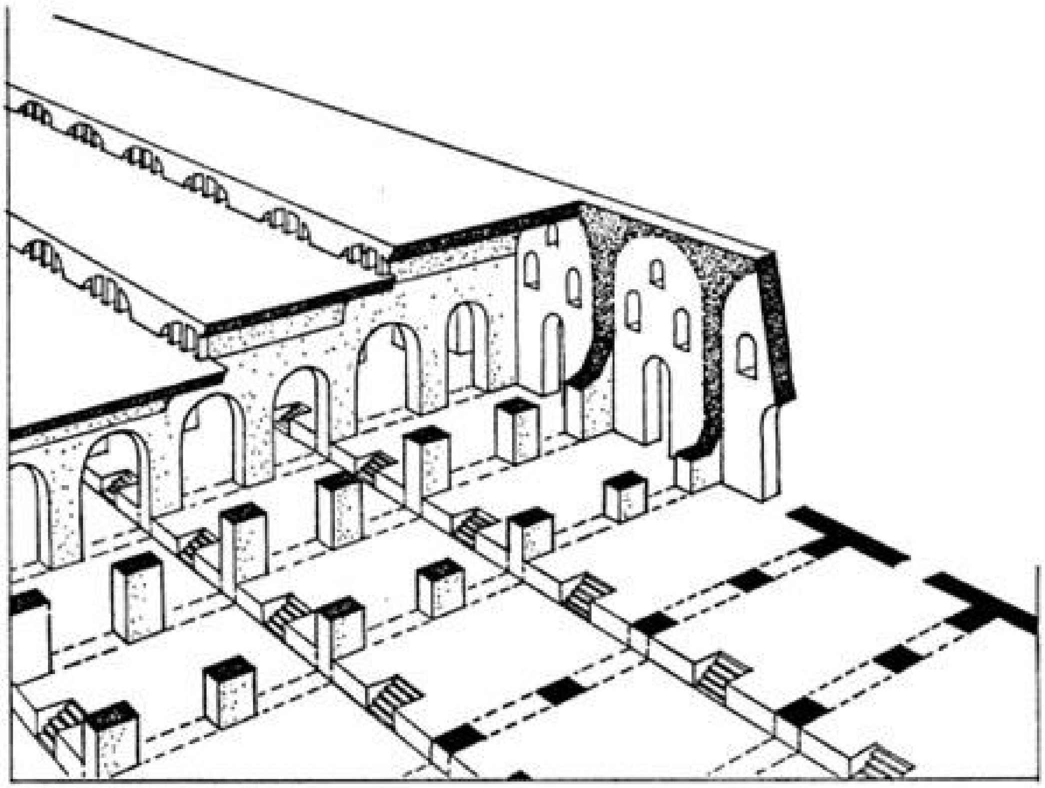

FIGURE 4. Cutaway of the PORTICUS ÆMILIA, southern Rome, early second century BCE (Sear 2002, p. 39).

3. THE ENGINEERING AND PHYSICS OF THE VAULTED FORM

Before discussing the further development of the vault in Rome, it is perhaps useful to present some background on the engineering and physics of the vault. The arch and the vaulted form have been so enduring primarily due to the physical superiority of such structures. From a physics standpoint, the arch facilitates structural integrity by redistributing the weight of the structure that it is supporting about the arced portion (in a line consistent with the said arc) and down through the piers, without inducing points of concentrated force as we see in corbelled or polygonal archways, or in simple post and lintel systems. The extruded arch, or barrel vault, the groin vault (formed by an intersection barrel vaults), and the dome (a revolved arch) all possess structural mechanics of a similar vein.

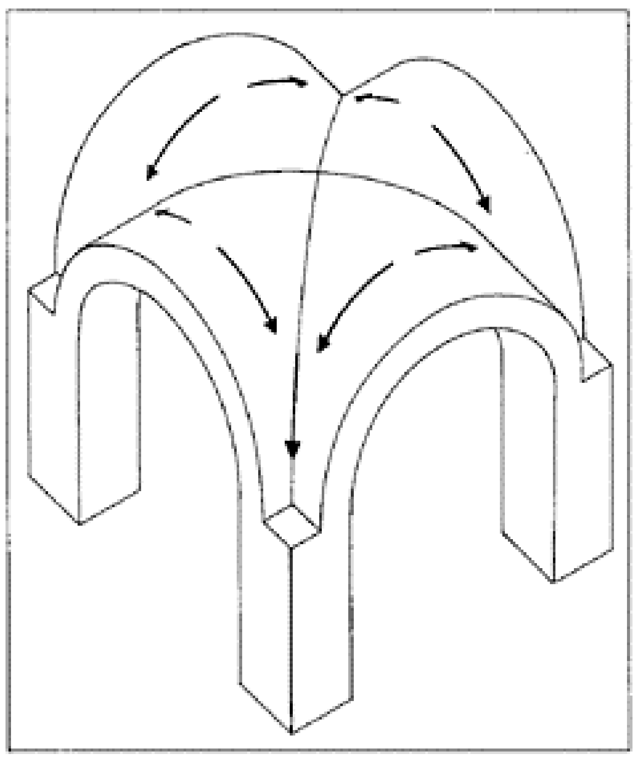

FIGURE 5. The stress distribution in a groin vault (Adam 2005, p. 395).

It is clear that the Romans did not possess a clear analytical understanding of the physics of the arch (this in fact would not happen until about a millennium after the fall of the Western Empire), although they did have a deep, intuitive understanding of the mechanics involved

(Lancaster 2005, p. 11). As is evidenced by the various structural deformations found at Præneste and elsewhere, the Romans tended to rely on on-site experimentation and development, as well as "accumulated knowledge" handed down from earlier generations, although a fundamental knowledge of geometry and structural mechanics (courtesy of Archimedes) was not lacking (Lancaster 2005, p. 10, p. 166).

3.1. **Formwork and Centering.** A tremendous amount of the success of any vault comes from the frame which is built to support it during the construction phase. The "centering" of a vault refers to the arced wooden frame which outlines the fundamental shape of the form, and which bears the weight of the masonry, or later, concrete laid upon it. The "formwork" of a vault is the portion of the support that sits upon the centering and upon which the concrete is actually placed, within a masonry facing at the perimeter of the formwork. It consists of numerous, straight beams joined together to approximate the circular shape of the vault. The centering and formwork are, after the locking in of the stonework or the curing of the concrete, removed; with large concrete vaults, this time period may be as long as two or three weeks (Lancaster 2005, p. 27; R. M. Taylor and R. Taylor 2003, p. 176). Occasionally, the formwork may be fitted with insets, so that coffers (stylized recesses designed to reduce overall weight) may be formed within the vaulted structure (R. M. Taylor and R. Taylor 2003, p. 188).

Just as with modern concrete building, there are usually clear traces of the formwork upon the final concrete structure (Lancaster 2005, p. 32; R. M. Taylor and R. Taylor 2003, p. 179). Due to the manner in which the centering interacts with the formwork, we can track the evolution of the centering; we see, throughout the course of Imperial Rome, Late Antiquity, and even into the Middle Ages, a better understanding of centering and of truss structures, resulting in less deformation of the vault during the curing process. The proper construction of the centering and the formwork constituted the bulk of vault construction, and, as mentioned before, usually occurred on-site with skilled carpenters playing an important role in the construction thereof (Lancaster 2005, p. 25-26). Moreover, removal of the formwork posed a significant challenge to the carpenters and engineers, and they had to be designed to anticipate this (R. M. Taylor and R. Taylor 2003, p. 188). The shape and the general layout of the formwork naturally depended greatly upon the sort of vaulted form being constructed, as illustrated below:

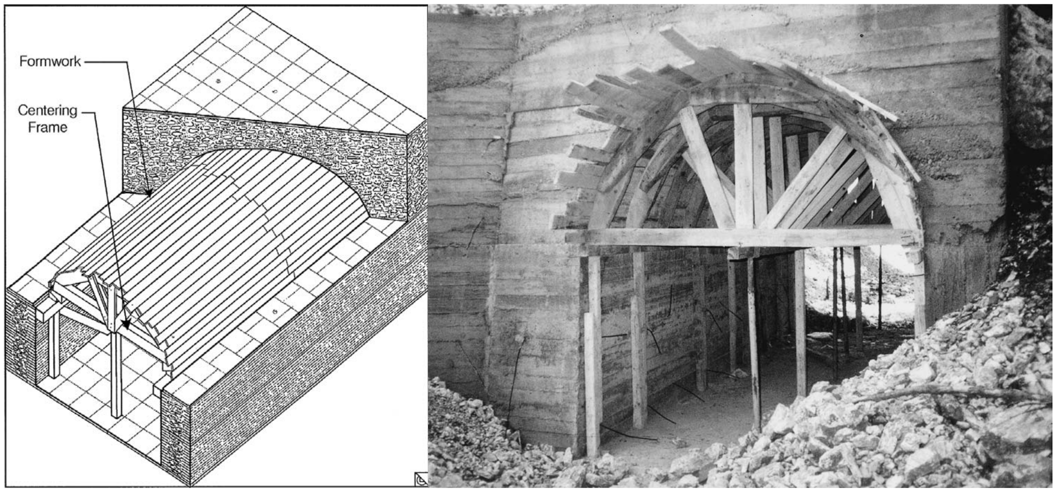

FIGURE 6. A schematic of the general centering and formwork plan for a barrel vault (left), and an example of modern centering and formwork nearly identical in form to the Roman method, Olympia, Greece (right) (Lancaster 2005, pp. 23 and 27).

4. The Monumentalization of the Vault

The TABULARIUM, or records office, was constructed in the FORUM ROMANUM on the front slope of the Capitoline Hill between 78 and 65 BCE, and demonstrates the foothold that concrete vaulted architecture had taken in central Rome by the middle of the first century BCE (Robertson 1929, p. 240; Sear 2002, p. 27; Lancaster 2005, p. 5). The TABULARIUM, which covered an area of approximately 230 feet by 140 feet, was full of arches, barrel vaults, and even primitive groin vaults formed by irregular intersections of barrel vaults (Robertson 1929, p. 240-241).

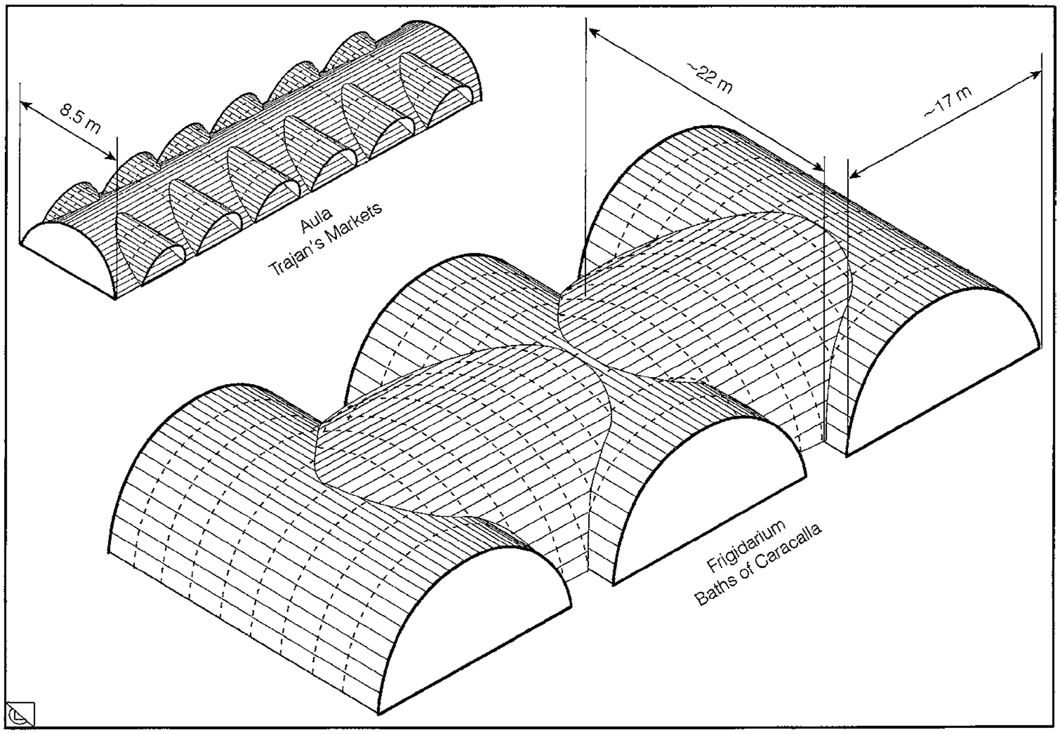

FIGURE 7. The formwork for the groin vaults at Trajan's Markets, 110 CE (upper left) and the FRIGIDARIUM at the Baths of Caracalla, 216 CE. As barrel vaults came to be intersected, the formwork naturally became much more complicated, and the potential for deformation greater (Lancaster 2005, p. 39).

Later, in this vein, we see the extension of concrete vaulting to structures such as the THEATRUM POMPEIUM (52 BCE) and the THEATRUM MARCELLI (12 BCE) (Lancaster 2005, p. 5; Ward-Perkins 1977, p. 64), where interior concrete vaults are combined with marble and stone architecture on the outside, indicating what Robertson refers to as a "divorce between function and decoration, due partly to the introduction of

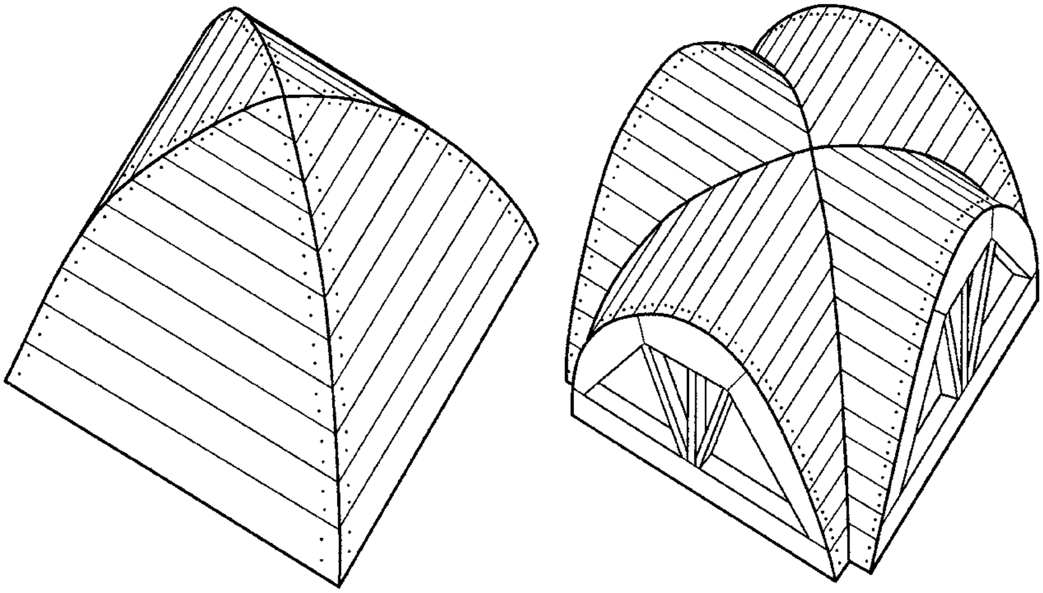

FIGURE 8. Formwork for two different types of groin vaults: the pavilion (left) and the cross (right) (Lancaster 2005, p. 38).

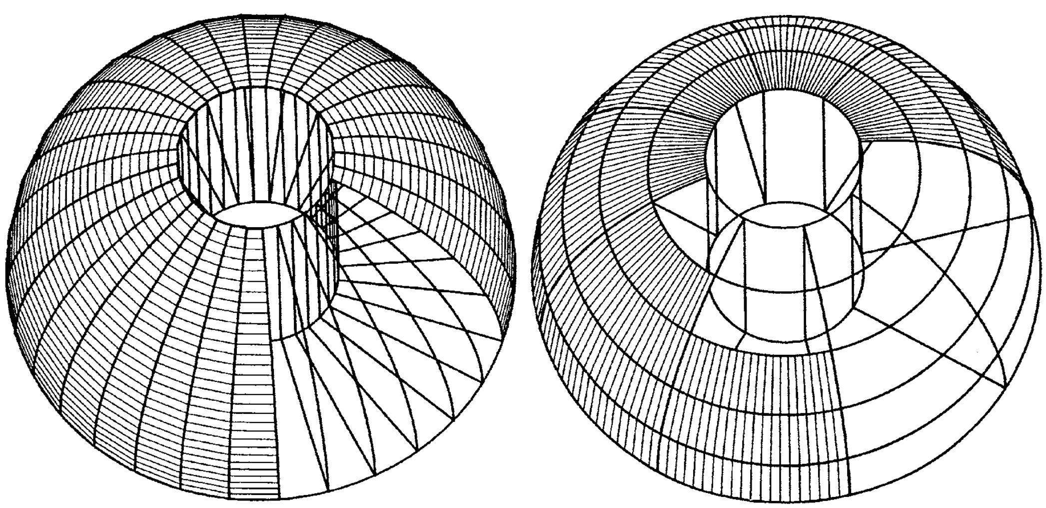

FIGURE 9. Two different styles of formwork for domes: the so called “horizontal” formwork (left) and the radial formwork (right) (Lancaster 2005, p. 40).

new structural methods, which is characteristic of Roman architecture" (Robertson 1929, p. 242).

In the Flavian period we see in the AMPHITHEATRUM FLAVIUM (completed 80 CE) and the DOMUS FLAVIA (completed 92 CE) a full and successful utilization of the cross or groin vault and the barrel vault on a colossal scale (Robertson 1929, p. 244; Lancaster 2005, p. 38). The former is particularly important, as it demonstrates a maturity of the forms initially investigated more than a century and a half earlier in the TABULARIUM.[3]

4.1. **The Groin Vault.** The groin vault is formed by the intersection of two barrel vaults, generally at right angles to one another (Adam 2005, p. 190). Amongst the earliest groin vaults we see are those in the Flavian Amphitheater. By the Trajanic period, the groin vault is in employ on a massive scale, as seen in the Markets of Trajan (110 CE) and the Baths of Trajan (109 CE). The great usefulness of the groin vault, especially as used in the roofing of Trajan's monuments, is that vaults that intersect along the axis perpendicular to the main length of the building can be used to allow light in. This is seen again in the Baths of Caracalla (see the example in Section 3.1), the Baths of Diocletian (ca. 300 CE), and in the Basilica of Maxentius (ca. 310 CE) (Kleiner 2016, p. 297).

[3]The AMPHITHEATRUM FLAVIUM also demonstrates the apparent "divorce" in form and function, as the building's outer casing of travertine is disjoint from the layered interior array of vaults, both barreled and crossed. This distinction, first seen in the TABULARIUM, becomes more evident in monuments such as the FORUM TRAIANI and Pantheon—both of the first quarter of the first century CE—which use concrete vaulted architecture with a distinct brickwork facing.

FIGURE 10. Groin Vault in the AMPHITHEATRUM FLAVIUM, 80 CE. Note the horizontal lines indicating original formwork. These lines are the often the only method for reconstructing formwork and centering, as there is no archaeological record of how they were actually employed (Lancaster 2005, p. 38).

Each of these massive structures, intended for mass public use, were well accommodated by the groin vaults and the light they provided.

4.2. **The Dome.** The earliest, preserved concrete dome of note is that belonging to the "Temple of Mercury" at BAIÆ in Campania, dating to the Augustan period. This particular dome surpassed the Tholos of Atreus at Mycenae in scale, and stood as the largest dome in existence until the construction of the Pantheon. The dome in general represents numerous challenges in construction, the greatest of which is likely the fabrication of the centering frame, which must consist of an array of the sort of frame required from a vault. Indeed, in the dome of the Temple of Mercury, there are numerous deformations to be found, a sign that the builders encountered substantial difficulties in shaping and positioning the frame for the large, 21 meter diameter dome (Lancaster 2005, p. 42).

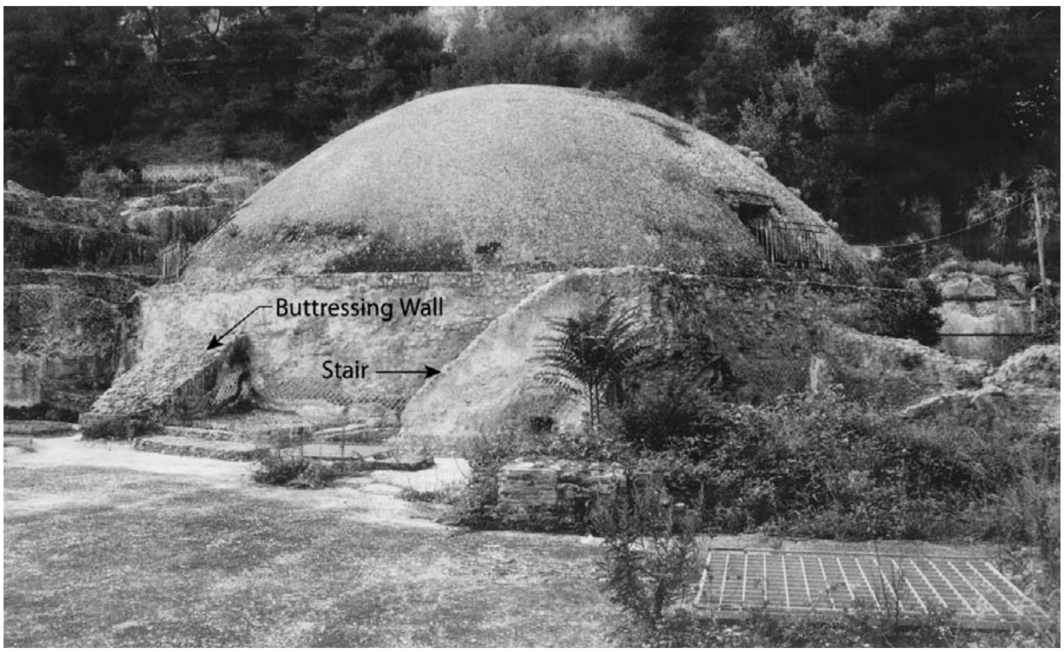

FIGURE 11. Temple of Mercury at BAIÆ, Augustan (Lancaster 2005, p. 143).

Nonetheless, there is evidence that the builders of the Temple of Mercury attempted to lighten the load that was borne by the dome by using less concrete (Sear 2002, p. 81). A similar thing would be done in the so-called "Octagonal Room" of Nero's Domus Aurea.[4] This pseudo-dome/polyhedral banquet hall, which represents one of the few surviving examples of "domed" architecture in the period between the Temple of Mercury and the Pantheon, dates to between 64 and 68 CE, and, as mentioned, uses a cement of lighter composition in order to reduce the weight of the domed structure. This would prove to be a precursor to the technique of "CÆMENTA gradation" that would be used to great effect in the Pantheon.

4.3. CÆMENTA Gradation. The selective grading of the CÆMENTA aggregate is one of the primary reasons for the lasting success of the Pantheon (Ward-Perkins 1977, p. 139). In short, "grading" means progressively using lighter aggregate material in the CÆMENTA at higher altitudes. From a physics standpoint (and mathematics aside), any non-vertical cross-section of a structure must support the weight (or downward acting gravitational force) of everything that sits above it. Consequently, the lowest portion of a structure sustains the greatest stress, while higher portions must bear less; the very top of a structure naturally bears no stress since nothing sits above it.

[4]The Octagonal Room of the DOMUS AUREA possesses many other interesting architectural features. Its well-preserved state shows, quite clearly, the formwork pattern upon the concrete, allowing for a fairly accurate reconstruction of the formwork used; the formwork, by this point, is of greater stability and certitude than that which was apparently used at BAIÆ, and which would further evolve into the methods employed in the Pantheon. The Octagonal Room's polyhedral design also removes the necessity of approximating an arced form by removing curvilinear geometries altogether.

Understanding this at least intuitively, the Romans spared using heavier (and hence stronger) materials towards the top of the Pantheon's 43.2 meter-diameter dome, since using heavier CÆMENTA would impose a greater stress burden on the lower portions of the dome. Moreover, since the higher portions of the dome would be supporting less weight anyway, it would be of no advantage to use a higher strength aggregate. The OCULUS that sits at the apex of the dome further reiterates this, by eschewing any structure altogether.

Grading seems to have been first used during the Augustan period (in the BASILICA ÆMILIA in the FORUM ROMANUM) and in the Neronian DOMUS AUREA, although it is not until the latter half of the first century CE that the practice becomes more commonplace (Lancaster 2005, p. 59). Prominent examples, aside from the Pantheon, that feature graded CÆMENTA include the Flavian Amphitheater, the BASILICA ARGENTARIA in the Forum of Caesar (early second century CE), and the Baths and Markets of Trajan, which immediately precede the Hadrianic Pantheon chronologically.

As far as the Pantheon is concerned, we see at the lowest levels of the dome, an aggregate composed of broken bricks, while higher up, a mixture of broken bricks and volcanic *tufo* stone is used. At the highest level, *tufo* and *scoria* (Vesuvian pumice) is the aggregate of choice (Lancaster 2005, p. 63). Consequently, the thickness of the dome sees a gradual thinning, from nearly 6 meters thick at the shoulder of the dome, to barely 150 centimeters at the oculus (Ward-Perkins 1977, p. 139).

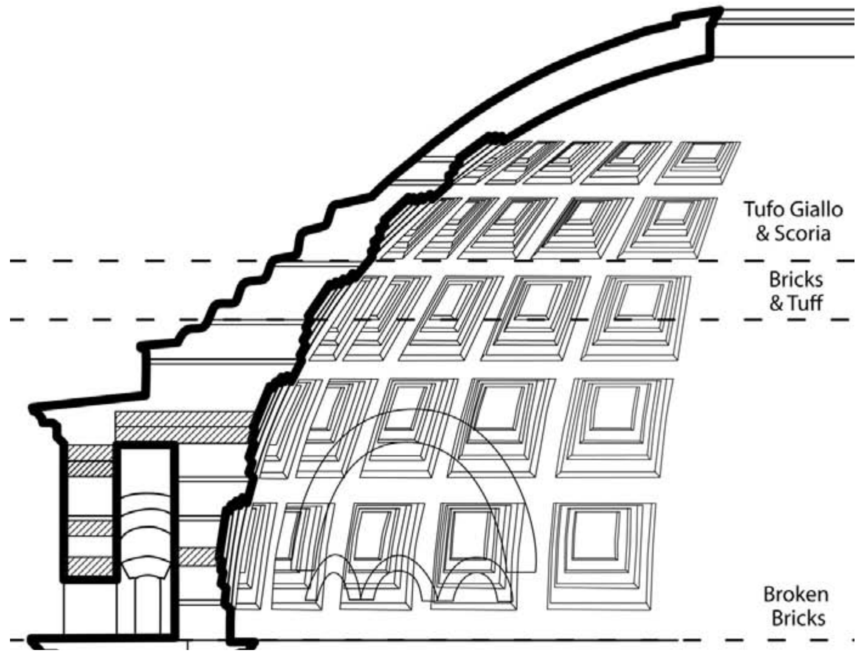

FIGURE 12. Gradation Scheme of the Pantheon's Dome, ca. 125 CE (Lancaster 2005, p. 62).

4.4. **Ribbed Vaulting and Internal Structural Supports.** "Ribbing" a vault refers to the practice of placing an arch within a concrete vaulted structure, ostensibly in order to relieve some of the weight situated directly above the said vault. The practice can first be seen in the polygonal walls at Alatri in Lazio and at the PORTA ROSA gate at *Velia* in Campania, both dating from about the third century BCE (Lancaster 2005, p. 87), the latter of which situates a stone *voussoir* rib directly above a true *voussoir* arch.

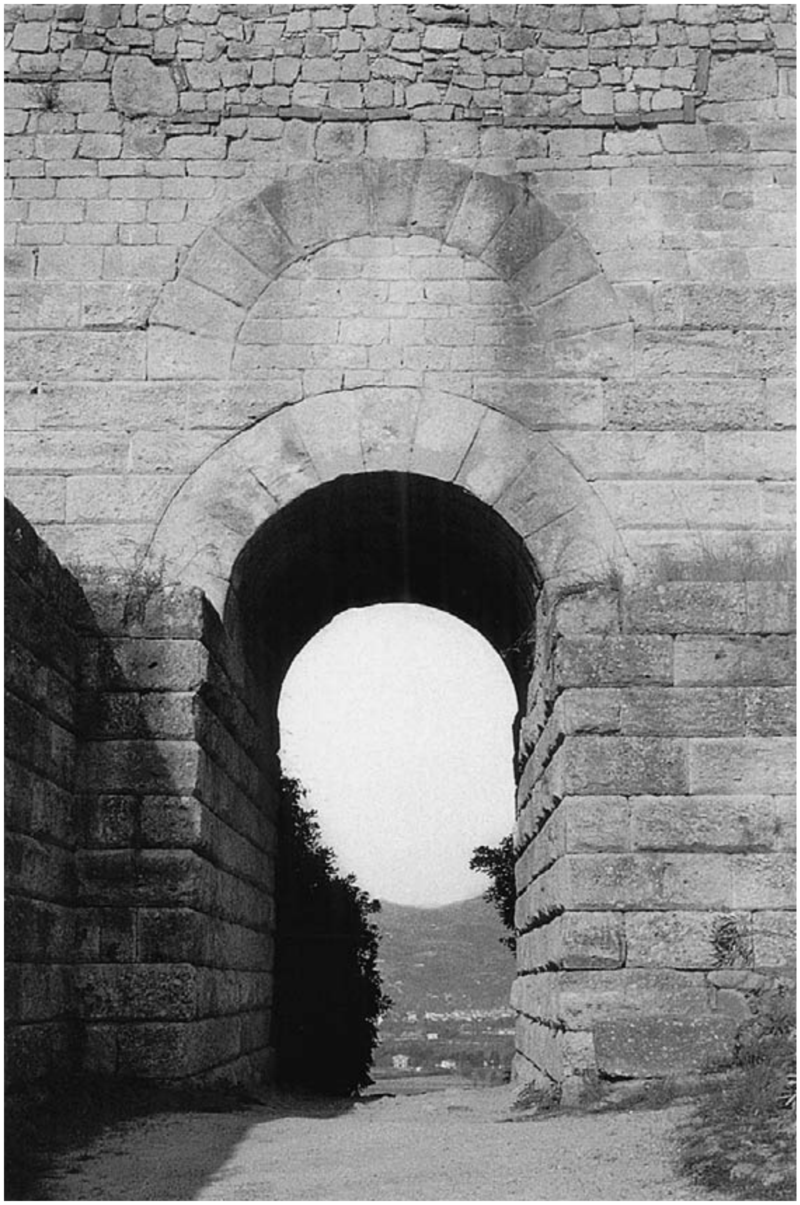

FIGURE 13. Ribbed vaulting, PORTA ROSA gate, *Velia*, third century CE (Lancaster 2005, p. 87).

The practice, in most cases, seems to serves no real structural purpose, since, if a rib is inserted into a solid wall, it no longer acts independently to divert loads (Ward-Perkins 1977, p. 152). Nonetheless, there is a general consensus that the ribs did help divert and manage the concrete which surrounded them. Ribbing is used extensively in the Flavian Amphitheater; in fact, the Colosseum, which features the earliest use of ribbing in Rome itself, utilizes several different kinds of ribbing based on the type of material comprising the rib.[5]

The Pantheon is unique in that it contains a staggered ribbing configuration around the rotunda wall that ultimately serves to dissect the rotunda walls into piers. This is done by placing a hollow, vertical shaft immediately below every other rib. In contrast to the placing of a rib into a solid facing, the system in place in the Pantheon features ribbing that is actually independent of the surrounding structure, acting, in the interior of the building, as a true arch.

Along similar lines, we see, first in the mid-second century CE and later with greater frequency in the fourth century CE, the use of amphorae (or *pignatte*, large earthenware jars) within concrete vaults. Evidence of these amphorae can be found today in the ruins of several monuments (most prominently in the Mausoleum of Helena, ca. 330 CE, whose Italian name, *Torpignattara*, takes its name from these jars) that

[5]The types of ribs used in the Flavian Amphitheater are travertine *voussoir*, brick-faced filled with concrete (the so-called ladder rib), and solid brick (or "BIPEDALIS").

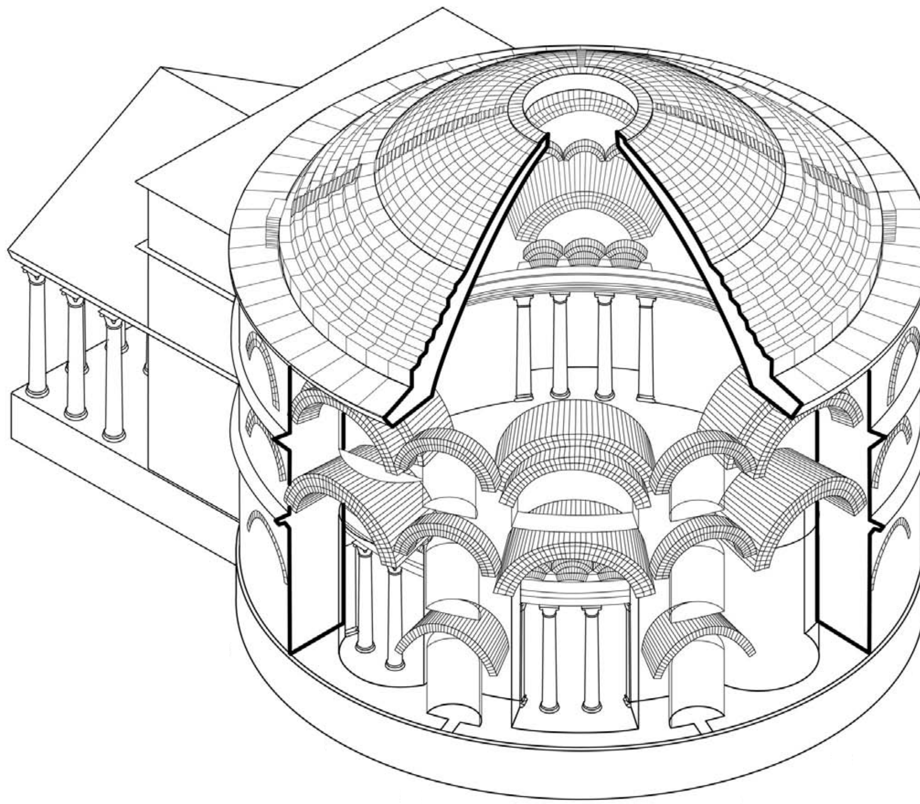

FIGURE 14. Ribbed vaulting array in the Pantheon, 125 CE. Note the hollows beneath each of the smaller ("minor") ribbed vaults, allowing each rib to function independent of the surrounding structure (Lancaster 2005, p. 100).

leave the amphorae exposed (Mason 1990). It is posited that the amphorae were used to reduce the weight of the vault, that they helped the curing of the concrete, and that they reduced the amount of other, more expensive materials used in the vaulting. Indeed, the amphorae were discarded ones, usually ones that contained fish or oil products that could not easily be cleaned and reused (Lancaster 2005, p. 69).

In a manner analogous to the spoliation demonstrated in Constantine's Arch, we see, in a somewhat less crass way, the reclamation of disregarded craftwork being used to save costs in a monumental project.

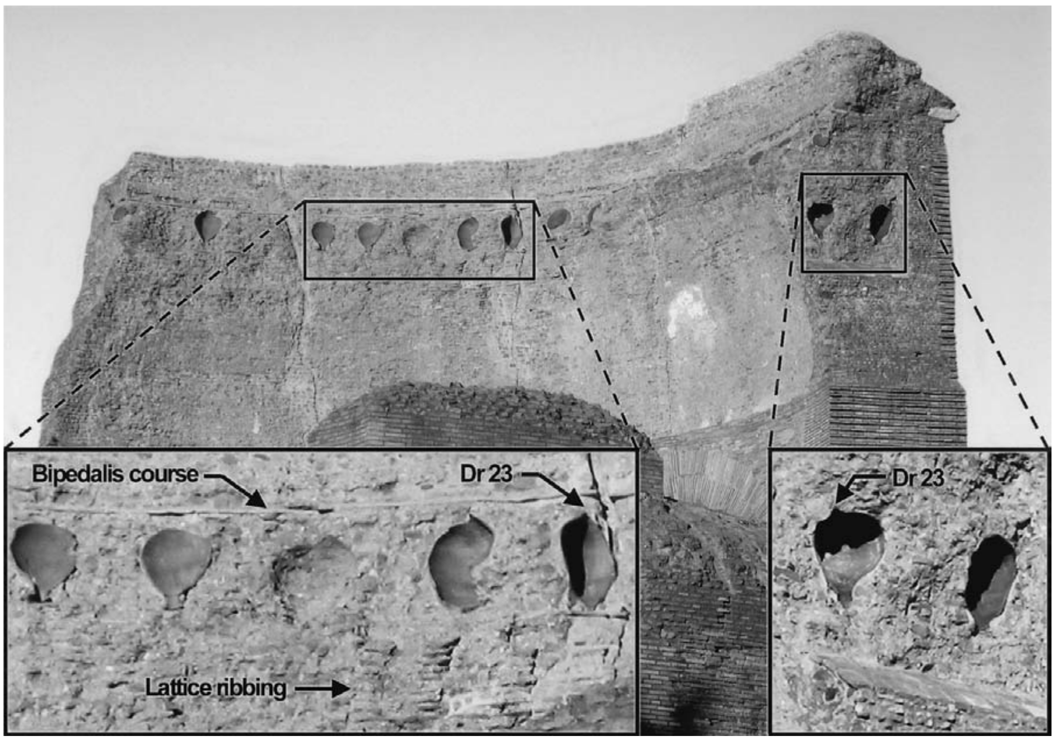

FIGURE 15. Exposed amphorae, as used at the Mausoleum of Helena, southeast Rome, ca. 330 CE (Lancaster 2005, p. 69).

Lastly, one architectural peculiarity, which was commonplace in Rome by the fifth century CE, deserves some mention. TUBI FITTILI, or vaulting tubes, were specially made earthenware tubes designed to interlock with one another and form an inner framework about which the concrete is laid. This technique, while increasing the weight of the vault minimally, allowed for an increase in the tensile strength of the vault, but also improved the insulation of the structure (Mason 1990).

5. Conclusion

The evolution of the vaulted form in Rome represents some 800 years worth of architectural development. From the earliest expressions in Italy of the true arch in Etruria and Campania in the fifth century BCE, we see the absorption of this form into Latin hands and mastery a thereof by the second century BCE. With the Roman development of *pozzolana*-based mortar, the concept of the arch was extended into barrel vaults, groin vaults, and eventually the dome. Moreover, each of these was developed on a massive scale, for mass public use. Indeed, without the perfection of the vault, many of the most long-lived of Rome's monuments would be non-existent or rendered useless. For without the groin vault, none of the great public bathhouses of the second century CE and beyond would contain enough light for the thousands of bathers within, and without the refinements in CÆMENTA gradation that eventually made their way into the Pantheon, the famous dome may well have collapsed long ago. Doubtless, vaulted architecture is one of Rome's more enduring legacies.

PART 2

THE MATHEMATICS OF ARCHES AND DOMES

1. Introduction

Here we derive the equations that characterize arches and axially-symmetric domes, and derive the solutions to those equations.

2. Arches

Let the shape of the arch be given by y. It is a function of x; we assume the arch's shape is symmetric about the line $x = 0$. At any given point in the arch's curve, we suppose the tension in that element of the arch is given by τ, and that it is directed along the arch element itself. The curve of the arch makes an angle ϕ with the horizontal at a position x, and an angle $\phi + \delta\phi$ at $x + \delta x$ (Fig. 16).

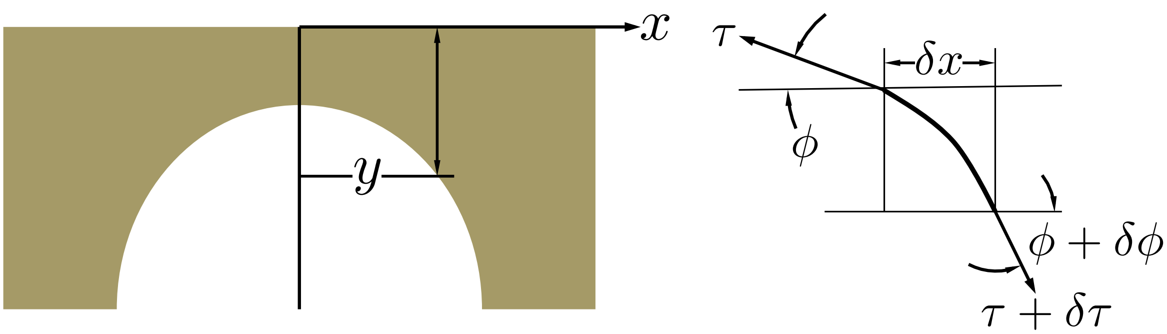

FIGURE 16. Geometry for modeling an arch.

The force balance in the vertical direction is then

$$\tau \sin\phi - \left(\tau + \frac{\mathrm{d}\tau}{\mathrm{d}x}\,\delta x\right)\left(\sin\phi + \cos\phi\,\frac{\mathrm{d}\phi}{\mathrm{d}x}\,\delta x\right) = f_y \delta x, \tag{1}$$

where f_y is the vertical force per unit length being exerted on the interior dome element. Expanding out the left hand side to first order and recombining terms gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\tau \sin\phi\right)\delta x = f_y \delta x. \tag{2}$$

In the horizontal direction, force balance yields

$$\tau\cos\phi - \left(\tau + \frac{\mathrm{d}\tau}{\mathrm{d}x}\,\delta x\right)\left(\cos\phi - \sin\phi\,\frac{\mathrm{d}\phi}{\mathrm{d}x}\,\delta x\right) = f_x \delta x, \tag{3}$$

where $f_x = 0$. Expanding out the left hand side to first order and recombining terms gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\tau \cos\phi\right)\delta x = 0. \tag{4}$$

This equation can immediately be integrated to yield

$$\tau\cos\phi\delta x = \mathcal{T}, \tag{5}$$

for a constant $\mathcal{T}$. $\mathcal{T}$ has units of force, and in fact $\mathcal{T}$ is the horizontal *thrust* which must be supported by the arch. Dividing Eq. (2) by Eq. (5), we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan\phi\right)\delta x = \frac{f_y}{\mathcal{T}}\delta x. \tag{6}$$

But $\tan\phi = \mathrm{d}y/\,\mathrm{d}x$, so that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\,\delta x = \frac{f_y}{\mathcal{T}}\delta x. \tag{7}$$

2.1. Thin, Self-Supporting Arch. For an arch that must support only its own mass, we have

$$f_y \delta x = \lambda g \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \delta x, \tag{8}$$

where λ is the mass per unit length of the arch material and g is the gravitational acceleration. The equation of shape is therefore

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\lambda g}{\mathcal{T}} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}. \tag{9}$$

The boundary conditions for this differential equation are

$$y(x = 0) = h, \tag{10}$$

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0} = 0 \tag{11}$$

where h is the maximum height of the arch.

2.2. Uniform Mass-Supporting Arch. For an arch that must support the weight of a uniform-density mass above it to a height y, we have

$$f_y \delta y = \rho g y \delta y \tag{12}$$

where ρ is the mass density of the material above the arch. The equation of shape becomes

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\rho g}{\mathcal{T}} y, \tag{13}$$

with boundary conditions

$$y(x = 0) = h, \tag{14}$$

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0} = 0. \tag{15}$$

3. DOMES

For domes, the geometry requires slightly more consideration. We assume it is axially-symmetric, with a profile defined by the curve $z(r)$, where r is measured from the axis of symmetry (Fig. 17).

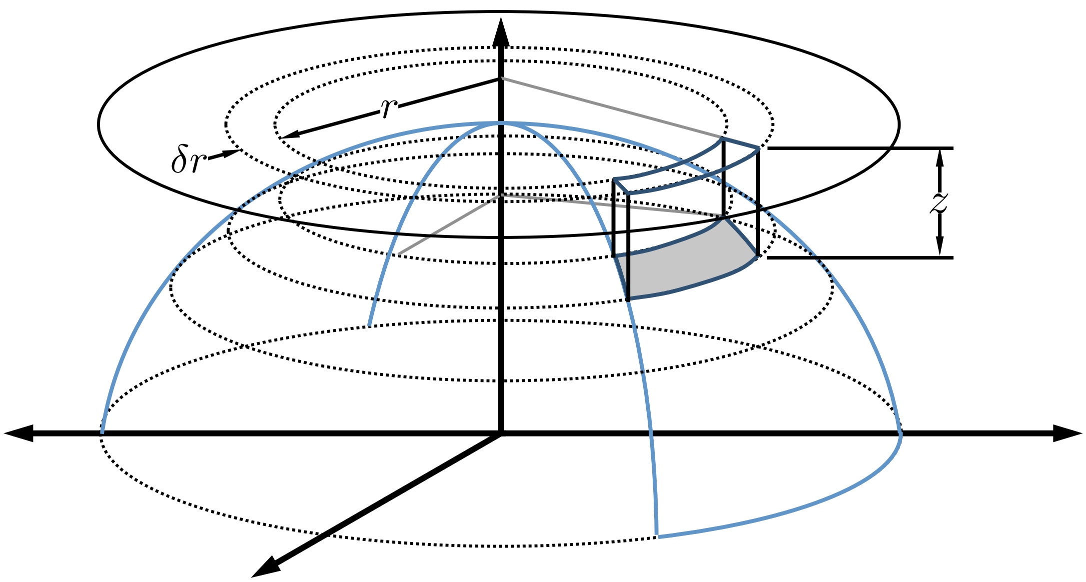

FIGURE 17. Geometry for modeling a dome.

The force balance in the vertical direction gives

$$\sigma r \sin\phi\delta\theta - \left(\sigma + \frac{\mathrm{d}\sigma}{\mathrm{d}r}\,\delta r\right)\left(\sin\phi + \cos\phi\,\frac{\mathrm{d}\phi}{\mathrm{d}r}\,\delta r\right)(r+\delta r)\delta\theta = f_z r\delta r\delta\theta, \quad (16)$$

where f_z is the vertical force per unit area being exerted on the interior dome element and σ is now an effective surface tension acting along the inner surface of the dome. Expanding out the left hand side to first order and recombining terms gives

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\sigma\sin\phi\right)\delta r\delta\theta = f_z r\delta r\delta\theta \quad (17)$$

In the horizontal direction, force balance yields

$$\sigma r \cos\phi\delta\theta - \left(\sigma + \frac{\mathrm{d}\sigma}{\mathrm{d}r}\,\delta r\right)\left(\cos\phi - \sin\phi\,\frac{\mathrm{d}\phi}{\mathrm{d}r}\,\delta r\right)(r+\delta r)\delta\theta = f_r r\delta r\delta\theta, \quad (18)$$

for a radial force per unit area $f_r \equiv 0$. Expanding to first order and recombining terms,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\sigma\cos\phi\right)\delta r\delta\theta = 0. \quad (19)$$

This equation can immediately be integrated to yield

$$r\sigma\cos\phi\delta r\delta\theta = \mathcal{T}; \quad (20)$$

again, $\mathcal{T}$ is the horizontal *thrust* which must be supported by the dome. Dividing Eq. (17) by Eq. (20), we get

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\tan\phi\right)\delta r\delta\theta = \frac{f_z}{\mathcal{T}} r\delta r\delta\theta. \quad (21)$$

However, $\tan\phi = \mathrm{d}z/\,\mathrm{d}r$, so that

$$\frac{\mathrm{d}^2 z}{\mathrm{d}r^2}\,\delta r\delta\theta = \frac{f_z}{\mathcal{T}} r\delta r\delta\theta. \tag{22}$$

3.1. **Thin, Self-Supporting Dome.** For a dome that must support only its own mass, we have

$$f_z r\delta r\delta\theta = \Sigma g r\sqrt{1+\left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2}\delta r\delta\theta, \tag{23}$$

where Σ is the mass per unit area of the dome material, assumed thin. The quantity

$$r\sqrt{1+\left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2}\delta r\delta\theta \tag{24}$$

is the differential surface area of the dome interior. The equation of shape is therefore

$$\frac{\mathrm{d}^2 z}{\mathrm{d}r^2} = \frac{\Sigma g}{\mathcal{T}} r\sqrt{1+\left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2} \tag{25}$$

The boundary conditions for this differential equation are

$$z(r=0) = h, \tag{26}$$

$$\left.\frac{\mathrm{d}z}{\mathrm{d}r}\right|_{r=0} = 0 \tag{27}$$

where h is the maximum height of the dome.

3.2. **Uniform Mass-Supporting Dome.** For a dome that must support a uniform-density mass above it to a height z, we have

$$f_z r \delta r \delta \theta = \rho g z r \delta r \delta \theta, \tag{28}$$

where ρ is the mass density of the material above the dome. The equation of shape becomes

$$\frac{\mathrm{d}^2 z}{\mathrm{d}r^2} = \frac{\rho g}{\mathcal{T}} z r. \tag{29}$$

The boundary conditions for this differential equation are also

$$z(r=0) = h, \tag{30}$$

$$\left.\frac{\mathrm{d}z}{\mathrm{d}r}\right|_{r=0} = 0. \tag{31}$$

4. Solutions

We derive here solutions for all four of the situations above.

4.1. **Arch Equations.** The arch equations Eqs. (9) and (13) admit elementary solutions. For the self-supporting arch, a straightforward change of variables gives

$$y = \sqrt{\frac{\lambda g}{\mathcal{T}}}\left[\cosh\left(\sqrt{\frac{\lambda g}{\mathcal{T}}}x\right) - 1\right] + h. \tag{32}$$

For the uniform mass-supporting arch, we have

$$y = h \cosh\left(\sqrt{\frac{\rho g}{\mathcal{T}}}x\right). \tag{33}$$

4.2. **Dome Equations.** Solutions for dome equations Eqs. (25) and (29) are more involved and can readily be evaluated numerically. However, analytic forms can be derived using standard functions.

First consider a general version of Eq. (25):

$$\frac{\mathrm{d}^2 z}{\mathrm{d}r^2} = k^2 r\sqrt{1 + \left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2}. \tag{34}$$

Multiply both sides by $(\mathrm{d}z/\,\mathrm{d}r)/2$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2\right] = \frac{k^2 r}{2}\left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)\sqrt{1 + \left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2}, \tag{35}$$

or

$$\frac{\mathrm{d}\phi}{\sqrt{\phi}\sqrt{1+\phi}} = \frac{k^2 r}{2}\,\mathrm{d}r, \tag{36}$$

where

$$\phi \equiv \left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2. \tag{37}$$

Using $\phi(r=0) = 0$, this can be integrated from $r = 0$ to r to give

$$\sqrt{\phi} = \sinh\left(\frac{k^2 r^2}{4}\right) = \frac{\mathrm{d}z}{\mathrm{d}r}\,. \tag{38}$$

This can be integrated once more to give

$$z(r) = h + \int_{r=0}^{r} \sinh\left(\frac{k^2 r'^2}{4}\right) \mathrm{d}r', \tag{39}$$

where for our problem

$$k = \sqrt{\frac{\Sigma g}{\mathcal{T}}}. \tag{40}$$

The integral here does have a closed form:

$$\int_{r=0}^{r} \sinh\left(\frac{k^2 r'^2}{4}\right) \mathrm{d}r' = \frac{\sqrt{\pi}}{4k}\left[\operatorname{erfi}\left(\frac{kr}{2}\right) - \operatorname{erf}\left(\frac{kr}{2}\right)\right], \tag{41}$$

where erfi and erf are the imaginary error function and error function, respectively. Thus, for the self-supporting dome,

$$z(r) = h + \sqrt{\frac{\pi\mathcal{T}}{16\Sigma g}}\left[\operatorname{erfi}\left(\sqrt{\frac{\Sigma g}{4\mathcal{T}}}r\right) - \operatorname{erf}\left(\sqrt{\frac{\Sigma g}{4\mathcal{T}}}r\right)\right]. \tag{42}$$

For the uniform mass-supporting dome, a general version of (29) reads

$$\frac{\mathrm{d}^2 z}{\mathrm{d}r^2} = m^2 z r. \tag{43}$$

This is immediately seen to be the Airy differential equation, for which two independent solutions are Ai and Bi, Airy functions of the first and second kind, respectively. A general solution to Eq. (43) without boundary conditions imposed reads

$$z(r) = C_1 \mathrm{Ai}\left(\sqrt[3]{m} r\right) + C_2 \mathrm{Bi}\left(\sqrt[3]{m} r\right), \tag{44}$$

for constants C_1 and C_2 to be determined. The boundary condition $\mathrm{d}z/\,\mathrm{d}r = 0$ at $r = 0$ yields

$$-\sqrt[3]{\frac{m}{3}}\frac{1}{\Gamma(\frac{1}{3})}C_1 + \sqrt[3]{m\sqrt{3}}\frac{1}{\Gamma(\frac{1}{3})}C_2 = 0, \tag{45}$$

so that

$$C_1 = \sqrt{3}C_2. \tag{46}$$

The boundary condition $z = h$ at $r = 0$ gives

$$C_2 = \frac{h}{2}3^{1/6}\Gamma\left(\frac{2}{3}\right). \tag{47}$$

Since here

$$m = \sqrt{\frac{\rho g}{\mathcal{T}}}, \tag{48}$$

we have for the uniform mass-supporting dome

$$z(r) = \frac{h}{2}3^{1/6}\Gamma\left(\frac{2}{3}\right)\left[\sqrt{3}\mathrm{Ai}\left(\sqrt[6]{\frac{\rho g}{\mathcal{T}}}r\right) + \mathrm{Bi}\left(\sqrt[6]{\frac{\rho g}{\mathcal{T}}}r\right)\right]. \tag{49}$$

5. Comparison

In Fig. 18, we show Eqs. (32), (33), (42), and (49), with the parameters in each problem chosen to limit the domain and range of each arch or dome to $[0, 1] \times [0, 1]$. In particular, we have chosen $h = 1$ for all cases, and

$$\sqrt{\frac{\lambda g}{\mathcal{T}}} = 1.2103, \quad \sqrt{\frac{\rho g}{\mathcal{T}}} = 1.3170,$$

$$\sqrt{\frac{\Sigma g}{4\mathcal{T}}} = 1.1509, \quad \sqrt[6]{\frac{\rho g}{\mathcal{T}}} = 1.7188.$$

In Fig. 19, we compare the two dome cases, visualizing them in three dimensions.

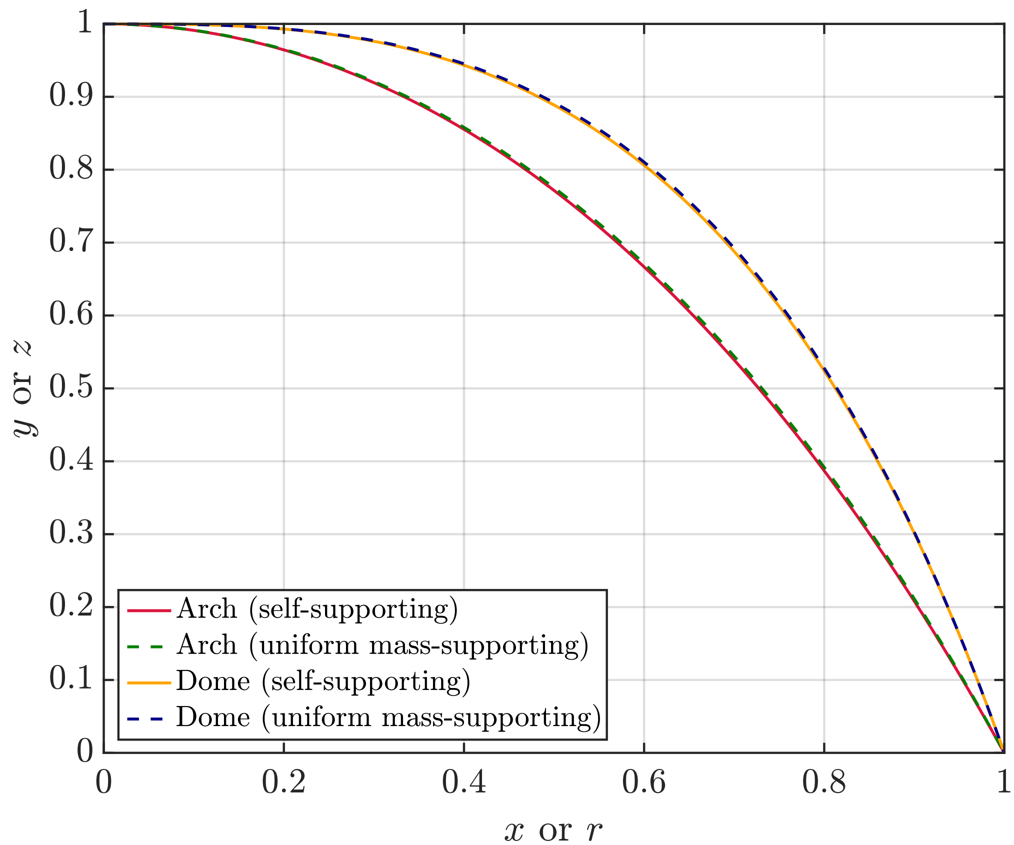

FIGURE 18. Comparison of self-supporting and uniformly supporting arch and dome shapes.

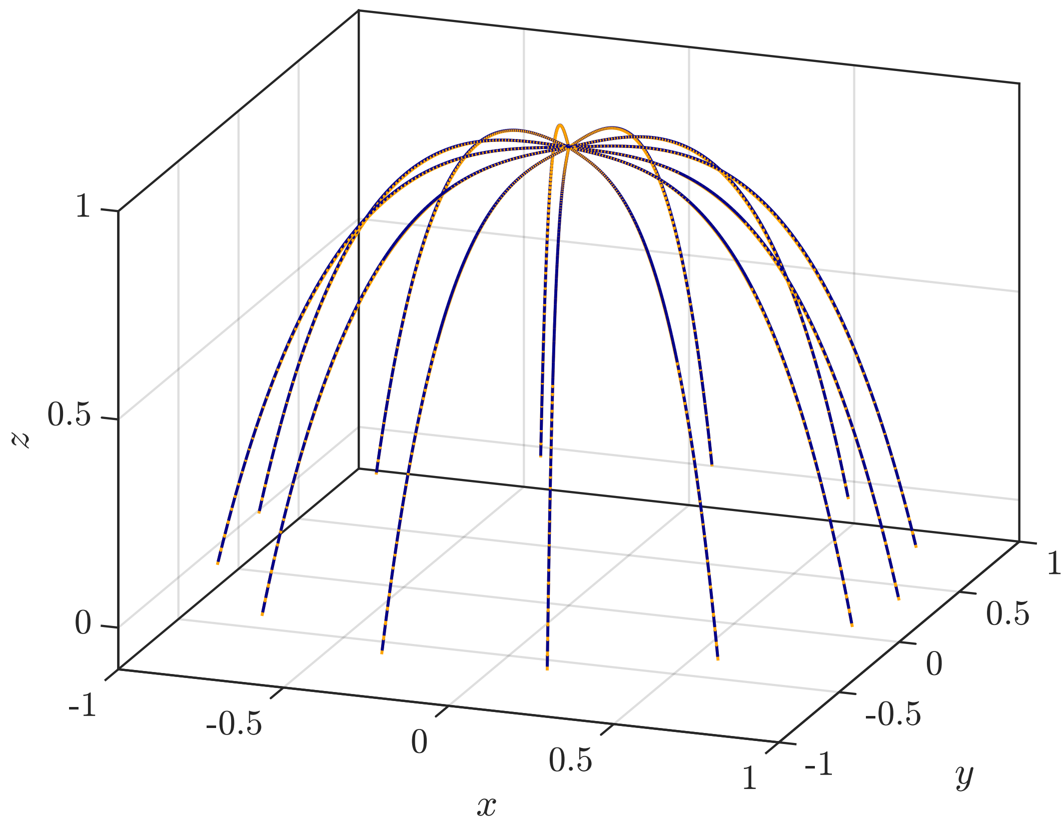

FIGURE 19. Comparison of self-supporting (solid orange line) and uniform mass-supporting (dashed blue line) dome shapes.

References

Adam, Jean-Pierre (2005). *Roman Building: Materials and Techniques*. Routledge.

Kleiner, Fred S (2016). *A History of Roman Art*. Cengage learning.

Lancaster, Lynne C (2005). *Concrete vaulted construction in Imperial Rome: innovations in context*. Cambridge University Press.

Mason, DJP (1990). "The Use of Earthenware Tubes in Roman Vault Construction: An Example From Chester". In: *Britannia* 21, pp. 215–222.

Robertson, Donald Struan (1929). *Greek and Roman Architecture*. Cambridge University Press.

Sear, Frank (2002). *Roman Architecture*. Routledge.

Taylor, Rabun M and Rabun Taylor (2003). *Roman Builders: A Study in Architectural Process*. Cambridge University Press Cambridge.

Ward-Perkins, John Bryan (1977). *Roman Architecture*. New York: HN Abrams.